A CHARACTERIZATION OF THE SYMMETRIC SQUARE OF A CURVE

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Abstract. In this paper a new intrinsic geometric characterization of the symmetric square of a curve and of the ordinary product of two curves is given. More precisely it is shown that the existence on a surface of general type S of irregularity q of an effective divisor D having self-intersection $D^2 > 0$ and arithmetic genus q implies that S is either birational to a product of curves or to the second symmetric product of a curve.
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1. Introduction

In this paper we give a new intrinsic geometric characterization of the symmetric square of a curve and of the ordinary product of two curves.

The symmetric products $S^k(C)$ of a curve C of genus $g > 0$ give, together with the ordinary products, the simplest examples of irregular varieties. To give some perspective, we recall that the cohomology ring of $S^k(C)$ approximates the cohomology ring of the Jacobian $J(C)$ of C. In particular one has ([M]) $H^1(C, \mathbb{Z}) \equiv H^1(J(C), \mathbb{Z}) \equiv H^1(S^k(C), \mathbb{Z})$ and, for $k > 1$,

$$H^2(S^k(C), \mathbb{Z}) \equiv H^2(J(C), \mathbb{Z}) \oplus \mathbb{Z}.$$

Notice that the Torelli-type theorem for $k < g$ (see [Ra2]) provides an equivalence between curves and symmetric products.

In view of their simple cohomological structure, one expects the symmetric products to have a very important place in the classification of irregular varieties. Some progress has been done using Fourier-Mukai transform and generic vanishing in the case $k = g-1$ by C. Hacon ([H]), giving a cohomological characterization of the theta divisor of a principally polarized abelian variety (PPAV).

For $1 < k < g-2$ there is a famous conjecture due to O. Debarre ([De2]) that claims that the symmetric products $S^k(C)$ and the Fano surface F of the lines of a smooth cubic 3-fold are the only varieties giving the minimal cohomological class of a PPAV (A, θ). The Debarre conjecture is proved for $q = 4$, by Z. Ran ([Ra1]), when $A = J(X)$ is a Jacobian of a curve by

Debarre ([De2]), and when A is the intermediate Jacobian of a generic cubic 3-fold by A. Höring ([Ho]).

Coming back to surfaces, it is a basic problem to give some geometrical or cohomological characterization of the second symmetric product of a curve.

In the spirit of surface classification theory, a conjecture, not equivalent, and in some sense even stronger than Debarre's, is the following:

Conjecture:
The only minimal surfaces X of irregularity $q > 2$ with $H^0(\Omega^2_X) \equiv \bigwedge^2 H^0(\Omega^1_X)$, are the symmetric products $S^2(C)$ and the Fano surfaces F of the lines of a smooth cubic 3-fold.

The above conjecture was proven for $q = 3$ in [HP] and independently in [Pi]. The proof of [HP] uses in a crucial way the fact that the image of the Albanese map is a divisor and it seems difficult to generalize. The proof of [Pi] is based on the geometric characterization of $S^2(C)$ given in [CCM] using the geometry of families of curves. Namely, in [CCM] $S^2(C)$ is proven to be the only minimal algebraic surface with irregularity q covered by curves of genus q and self-intersection 1.

Analysing the curves of small genus on a surfaces of general type, we discovered a surprisingly precise geometric characterization of the second symmetric product (and of the ordinary product).

Theorem 1.1. *Let S be a smooth surface of general type with irregularity q containing a 1-connected divisor D such that $p_a(D) = q$ and $D^2 > 0$. Then the minimal model of S is either:*

(a) the product of two curves of genus $g_1, g_2 \geq 2$ (and $g_1 + g_2 = q$) or
(b) the symmetric product $S^2(C)$ where C is a smooth curve of genus q (and $C^2 = 1$).

Furthermore, if D is 2-connected, only the second case occurs.

This result is in some sense very atypical of the theory of algebraic surfaces, because we obtain a complete classification of the surface from the existence of a single divisor with certain properties. The only similar instance we know of is the characterization of rational surfaces from the existence of a smooth rational curve with positive self-intersection.

The proof of Theorem 1.1 consists of two main steps, that we describe in the symmetric product case. First we reduce to the case when the curve C is smooth. This step requires a very careful numerical analysis of the effective divisors contained in C. Then we observe that the Albanese variety $\mathrm{Alb}(X)$ of the surface is isomorphic to the Jacobian $J(C)$ of C and, combining the Brill-Noether theory on C with the generic vanishing theorem of Green and Lazarsfeld, we show that the curve C moves in a positive dimensional family. This allows us to use the results in [CCM] and complete the proof.

It appears that a sort of weak Brill-Noether theory can be performed for line bundles on irregular surfaces. We will come back to these topics in a forthcoming paper (cf. [MPP3]).

Notation: A *surface* is a smooth projective complex surface. The *irregularity* of a surface S, often denoted by q or $q(S)$, is $h^1(\mathcal{O}_S) = h^0(\Omega^1_S)$ and the *geometric genus* $p_g(S)$ is $h^0(K_S) = h^2(\mathcal{O}_S)$. A *curve* on a surface is a nonzero effective divisor. A *fibration* of genus b of a surface S is a map $f\colon S \to B$ with connected fibers, where B is a smooth curve of genus b; f is *relatively minimal* if its fibers contain no -1-curve. A *quasi-bundle* is a fibration such that all the smooth fibers are isomorphic and the singular fibers of are multiples of smooth curves.

2. Auxiliary facts

In this section we collect some auxiliary facts. We start by recalling the well known equality for reducible curves on smooth surfaces:

• If a curve D decomposes as $D = A + B$ where $A, B > 0$ and $AB = m$ then $p_a(A) + p_A(B) + m - 1 = p_a(D)$.

The following results are needed in the sequel:

Lemma 2.1. *Let S be a surface and let $f\colon S \to B$ a relatively minimal fibration of genus ≥ 2 with general fibre of genus ≥ 2. If C is a section of f (i.e. an irreducible curve C such that $CF = 1$), then:*

(i) *$C^2 \leq 0$;*
(ii) *if $C^2 = 0$, then f is a quasi-bundle and there exists $m > 0$ such that mC is a fiber of a quasi-bundle fibration of S.*

Proof. (i) Let us notice first that any section C of f is smooth of genus b. Denote by F the general fiber of f; by Arakelov's theorem (cf. [Se, Theorem 3.1]) the relative canonical class $K_{S/B} \sim K_S - (2b-2)F$ is nef, hence $K_S C \geq (2b-2)$ and, by the adjunction formula, $C^2 \leq 0$.

(ii): follows by Theorem 3.2 in [Se]. □

Lemma 2.2. *Let D be a 1-connected curve on a surface. Then every subcurve $A < D$ satisfies $p_a(A) \leq p_a(D)$.*

Proof. Set $B := D - A$ and let $m := AB$. Then $p_a(A) + p_a(B) + m - 1 = p_a(D)$. Suppose for contradiction that $p_a(A) > p_a(D)$. Then we obtain $p_a(B) + m - 1 < 0$, i.e. $m < 1 - p_a(B)$. Since $1 - p_a(B) = h^0(B, \mathcal{O}_B) - h^1(B, \mathcal{O}_B)$ we conclude that $m < h^0(B, \mathcal{O}_B)$. This contradicts [KM, Lemma 1.4], that states that any subcurve D' of a 1-connected curve D satisfying $D'(D - D') = b$ verifies $h^0(D', \mathcal{O}_{D'}) \leq b$.

□

Lemma 2.3. *Let S be a surface of general type and let D be an irreducible curve of S. If $D^2 > 0$, then:*

(i) *$p_a(D) \geq 2$;*

(ii) *if* $p_a(D) = 2$, *then the minimal model* T *of* S *has* $K_T^2 = 1$, $q(T) = 0$.

Proof. Let $\eta\colon S \to T$ be the morphism onto the minimal model, so that $K_S := \eta^* K_T + E$, where E is the exceptional divisor. Since $D^2 > 0$, the curve D is not contracted by η, hence $K_S D \geq \eta^* K_T D > 0$ and the adjunction formula gives immediately $p_a(D) \geq 2$, with equality holding if and only if $D^2 = \eta^* K_T D = 1$. Hence, if $p_a(D) = 2$ the index theorem gives $K_T^2 = 1$ and, since by [De1] irregular surfaces have $K^2 \geq 2p_g$, T is regular. □

3. Curves with $p_a = q$

As a preparation for the proof of Theorem 1.1, in this section we make a detailed numerical analysis of the following situation:

- S is a surface of general type with irregularity q;
- D is a 1-connected curve of S such that $p_a(D) = q$ and $D^2 > 0$.

Proposition 3.1. *Let* S *be a surface of general type with irregularity* q *and let* D *be a* 1*-connected curve of* S *such that* $p_a(D) = q$ *and* $D^2 > 0$. *If* $D = A + B$, *where* A, B *are curves such that* $AB = 1$ *and* $p_a(A) \geq 1$, $p_a(B) \geq 1$, *then* S *is birational to the product of two curves.*

Proof. The equality $AB = 1$ implies that both A and B are 1-connected (see [CFM, Lemma A.4]) and that $q = p_a(D) = p_a(A) + p_a(B)$. Since, by assumption, $p_a(A) \geq 1$, $p_a(B) \geq 1$, one has also $p_a(A) < q$ and $p_a(B) < q$. So both $A^2 \leq 0$, $B^2 \leq 0$ (cf. [R], [Ca, Remark 6.8] and also [MPP2]). By the hypothesis $D^2 = A^2 + 2 + B^2 > 0$, at least one of the inequalities is in fact an equality. Suppose then that $B^2 = 0$. Then (cf. ibidem) there exists a fibration $f\colon S \to E$ where E is a curve of geometric genus $g(E) \geq q - p_a(B)$ and such that mB is a fibre F of f for some $m > 0$.

Since $AB = 1$ and B is nef, there is a unique irreducible curve $\theta \leq A$ such that $\theta B \neq 0$. Of course θ is not contained in a fibre of f and, by Lemma 2.2, $p_a(\theta) \leq p_a(A)$. From $p_a(A) = q - p_a(B)$ we obtain $p_a(\theta) \leq g(E)$. Since f induces a surjective morphism $\theta \to E$ of degree m and $p_a(\theta) \leq g(E)$, we conclude, by the Hurwitz formula, that $g(\theta) = p_a(\theta) = g(E)$ and, in addition, $m = 1$ or $g(E) = 1$ and $\theta \to E$ is unramified. In the latter case, if $m > 1$ we have a contradiction because the existence of a multiple fibre of f means that the cover is ramified. So $m = 1$ and the fibration $f\colon S \to E$ satisfies $g(E) + g(F) = q$. By [Be, Lemme], S is birational to the product $F \times E$. □

Next we show that, when S is not birational to the product of two curves, we can assume that D is 2-connected.

Lemma 3.2. *Let* $\tilde{S}$ *be an irregular surface of general type that is not birational to the product of two curves. Let* D *be a* 1*-connected curve of* $\tilde{S}$ *such that* $p_a(D) = q$, $D^2 > 0$. *Then there is a birational morphism* $p\colon \tilde{S} \to S_0$, *where* S_0 *is a smooth surface such that* $p(D)$ *contains a* 2*-connected curve* D_0 *satisfying* $D_0^2 > 0$ *and* $p_a(D_0) = q$.

Proof. If D is 2-connected there is nothing to prove. Otherwise we have a decomposition $D = A + B$ where $AB = 1$ and, by Lemma 3.1, $p_a(A) = 0$, $p_a(B) = q$. By [CFM, Lemma A.4], A and B are 1-connected. Since $p_a(A) = 0$, A is contracted by the Albanese map and so $A^2 < 0$.

If $A^2 < -1$, then from $D^2 > 0$ and $AB = 1$ we must have $B^2 > 0$ and we consider now the curve B. If B is 2-connected we take $D_0 := B$. If B is not 2-connected, then it has a decomposition $B = A_1 + A_2$ with $A_1A_2 = 1$. As above we can suppose $p_a(A_1) = 0$ and $p_a(A_2) = q$ and we can now restart the reasoning.

If $A^2 = -1$, then A contains an irreducible (-1)-curve E such that $ED = 0$. If $p\colon S \to S_1$ is the blow down of E and $D_1 := p(D)$, then we have $D = p^*D_1$, hence D_1 is 1-connected, $D_1^2 = D^2 > 0$ and $p_a(D_1) = p_a(D) = q$. Hence we may replace D and S by D_1 and S_1 and, if D_1 is not 2-connected, repeat the previous step.

Since D has a finite number of components, the process described above must stop and so in the end we get a surface S_0 birational to S contained a 2-connected D_0 as wished. □

4. The 2-connected case

Here we examine the situation of §3 under the additional assumption that D be 2-connected.

Lemma 4.1. *Let S be an irregular surface of general type and let D be a 2-connected curve of S such that $p_a(D) = q$ and $D^2 > 0$. Then:*

(i) *D is contained in the fixed part of $|K_S + D|$;*
(ii) *D is smooth irreducible;*
(iii) *$h^0(S, D) = 1$;*
(iv) *the Albanese image of S is a surface.*

Proof. (i) Since $D^2 > 0$ and, because D is 1-connected, $h^0(D, \mathcal{O}_D) = 1$, one has $h^1(S, K_S + D) = 0$. Hence the cokernel of the restriction map $H^0(S, K_S + D) \to H^0(D, \omega_D)$ is $H^1(S, K_S)$ and therefore by $p_a(D) = q$, the image of r must be zero.

(ii) Assume by contradiction that D is not smooth. Then D has multiple points. Since D is contained in the fixed part of $|K_S + D|$, any multiple point P of D is a base point of $|K_S + D|$ and so, by [ML, Thm. 3.1], D is not 2-connected. This contradicts the hypothesis. Finally, D, being smooth and 1-connected, is necessarily irreducible.

(iii) Assume that $h^0(S, D) > 1$. Then the irreducible curve D moves. Since, by hypothesis, $p_g(S) > 0$, we conclude that D is not a fixed component of $|K_S + D|$, contradicting (i).

(iv) Since D is irreducible we have $q \geq 3$ by Lemma 2.3. Assume for contradiction that the Albanese image of S is a curve E. Then E is a smooth curve of genus q and we have thus a fibration $f\colon S \to E$. Since the smooth curve D satisfies $D^2 > 0$, D is not contained in any fibre F of f, and

so f induces a degree m morphism $f|_D\colon D \to E$, where $m := DF$. Since $g(D) = g(E) = q$ and $q \geq 3$, we have $m = 1$ by the Hurwitz formula.

Now notice that we can assume that S is minimal, and thus f is relatively minimal. In fact every (-1)-curve θ is contained in the fibres of S and thus, since $DF = 1$, the image of D in the minimal model of S is still a smooth curve with geometric genus q. Then D is a section of f, but this contradicts Lemma 2.1.

So the Albanese image of S must be a surface. □

Corollary 4.2. *Let S be an irregular surface of general type that is not birational to a product of curves and let D be a nef 2-connected curve of S such that $p_a(D) = q$ and $D^2 > 0$. Assume also that there is no (-1)-curve θ such that $D\theta = 0$. Then any curve C numerically equivalent to D is smooth irreducible.*

Proof. Since D is nef, also C is nef. Then it is well known that C nef and big implies that C is 1-connected (see [ML, Lemma 2.6]). If C is not 2-connected there is a decomposition $C = A + B$ where $AB = 1$ and, by Proposition 3.1, $p_a(A) = 0$, $p_a(B) = q$. As in Lemma 3.2 one has $A^2 < 0$ and since C is nef one must have $A^2 = -1$, but this contradicts the hypothesis on D. Hence C is 2-connected and so, by Lemma 4.1, C is smooth irreducible.

□

Proposition 4.3. *Let S be a surface of general type with irregularity q and let D be a smooth irreducible curve D such that $d := D^2 > 0$ and $g(D) = q$. Then the set $\{L \in \mathrm{Pic}^0(S) | h^0(D + \eta) > 0\}$ has dimension $\geq \min\{q, d\}$.*

Proof. By Lemma 4.1, the Albanese image of S is a surface. So, by the generic vanishing theorem of Green-Lazarsfeld ([GL1],[GL2]), the set $V^1(S) = \{\eta \in \mathrm{Pic}^0(S) | h^1(\eta) > 0\}$ is the union of finitely many translates of proper abelian subvarieties of A. Since $D^2 > 0$, the map $\mathrm{Pic}^0(S) \to \mathrm{Pic}^0(D)$ is injective and thus an isomorphism, hence we can identify $\mathrm{Pic}^0(S)$ and $\mathrm{Pic}^0(D)$.

Denote by $W_d(D) \subset \mathrm{Pic}^0(D)$ the image of the natural map $S^d(D) \to \mathrm{Pic}^0(D)$ defined by $\Delta \mapsto [\mathcal{O}_D(\Delta - D)]$. If $d \geq q$ then $W_d(D) = \mathrm{Pic}^0(D)$, otherwise $W_d(D)$ is d-dimensional. Then $W_d(D)$ generates $\mathrm{Pic}^0(D)$ and it is irreducible, hence it cannot be contained in $V^1(S)$.

Note that for $\eta \notin V^1(S)$ the restriction sequence $0 \to \eta \to \eta + D \to (\eta + D)|_D \to 0$ gives an isomorphism $H^0(\eta + D) \cong H^0((\eta + D)|_D)$. Hence for every $\eta \in W_d(D) \setminus V^1(S)$ (and, by semicontinuity, for every $\eta \in W_d(D)$) we have $h^0(D + \eta) > 0$. □

5. Proof of the main result

Proof of Theorem 1.1. By Proposition 3.1, if there exists a decomposition $D = A + B$, with $A, B > 0$, $AB = 1$ and $p_a(A), p_a(B) > 0$, then S is birational to a product of curves, namely we have case (a).

So assume that no such decomposition exists. Then, up to replacing S by a surface in the same birational class, by Proposition 3.2 we may assume that D is 2-connected, hence smooth and irreducible by Lemma 4.1.

Write $d := D^2$. By Proposition 4.3, there exist a d-dimensional system $\mathcal{C}$ of curves numerically equivalent to D. Since we can obviously assume that there is no (-1)-curve A such that $DA = 0$, any curve C of $\mathcal{C}$ is smooth by Lemma 4.2. The Jacobian of every curve of $\mathcal{C}$ is isomorphic to $\mathrm{Pic}^0(D)$, hence the smooth elements of $\mathcal{C}$ are all isomorphic to D.

If $d > 1$, by [GP, Lemma 2.2.1] (cf. also [CCM, §0]) S is not of general type, against the assumptions. So we have $d = 1$ and the result follows by [CCM, Thm. 0.20]. □

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